Two results on the digraph chromatic number

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Abstract

It is known (Bollobás [4]; Kostochka and Mazurova [12]) that there exist graphs of maximum degree Δ and of arbitrarily large girth whose chromatic number is at least $c\Delta/\log\Delta$. We show an analogous result for digraphs where the chromatic number of a digraph D is defined as the minimum integer k so that $V(D)$ can be partitioned into k acyclic sets, and the girth is the length of the shortest cycle in the corresponding undirected graph. It is also shown, in the same vein as an old result of Erdős [5], that there are digraphs with arbitrarily large chromatic number where every large subset of vertices is 2-colorable.

Keywords: Chromatic number, digraph, digraph coloring, dichromatic number, girth.

1 Digraph Colorings

Let D be a (loopless) digraph. A vertex set $A \subset V(D)$ is called *acyclic* if the induced subdigraph $D[A]$ has no directed cycles. A *k-coloring* of D is a partition of $V(D)$ into k or fewer acyclic sets. The minimum integer k for which there exists a k-coloring of D is the *chromatic number* $\chi(D)$ of the

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digraph D. This definition of the chromatic number of a digraph was first treated by Neumann-Lara [16]. The same notion was independently introduced two decades later when considering the circular chromatic number of weighted (directed or undirected) graphs [14], and further treated in [3].

This notion of colorings of digraphs turns out to be the natural way of extending the theory of undirected graph colorings since it provides extensions of most of the basic results from graph coloring theory [3, 7, 8, 14, 15].

In this note we prove, using standard probabilistic approach, that two further analogues of graph coloring results carry over to digraphs. The first result, see Theorem 2.1, provides evidence that the digraph chromatic number, like the graph chromatic number, is a global parameter that cannot be deduced from local considerations. The second result, see Theorem 3.1, shows that there are digraphs with large chromatic number k in which every set of at most $c|V(D)|$ vertices is 2-colorable, where $c > 0$ is a constant that only depends on k. The analogous result for digraphs was proved by Erdős [5] with its outcome being that all sets of at most cn are 3-colorable. Both the 3-colorability in Erdős' result and 2-colorability in Theorem 3.1 are best possible.

Concerning the first result, it is well-known that there exist graphs with large girth and large chromatic number. Bollobás [4] and, independently, Kostochka and Mazurova [12] proved that there exist graphs of maximum degree at most Δ and of arbitrarily large girth whose chromatic number is $\Omega(\Delta/\log\Delta)$. Our Theorem 2.1 provides an extension to digraphs.

The bound of $\Omega(\Delta/\log\Delta)$ from [4, 12] is essentially best possible: a result of Johansson [10] shows that if G is triangle-free, then the chromatic number is $O(\Delta/\log\Delta)$. Similarly, Theorem 3.1 is also essentially best possible: Erdős et al. [6] showed that every tournament on n vertices has chromatic number $O(\frac{n}{\log n})$.

In general, it may be true that the following analog of Johansson's result holds for digon-free digraphs, as conjectured by McDiarmid and Mohar [13].

Conjecture 1.1. *Every digraph D without digons and with maximum total degree Δ has $\chi(D) = O(\frac{\Delta}{\log\Delta})$.*

Theorem 2.1 shows that Conjecture 1.1, if true, is essentially best possible.

2 Chromatic number and girth

First, we need some basic definitions. For an extensive treatment of digraphs, we refer the reader to [2]. Given a loopless digraph D, a *cycle* in D

is a cycle in the underlying undirected graph. The *girth* of D is the length of a shortest cycle in D, and the *digirth* of D is the length of a shortest directed cycle in D. The *total degree* of a vertex v is the number of arcs incident to v. The *maximum total degree* of D, denoted by $\Delta(D)$, is the maximum of all total degrees of vertices in D. The *out-degree* and the *in-degree* of a vertex v are denoted by $d^+(v)$ and $d^-(v)$, respectively.

It is proved in [3] that there are digraphs of arbitrarily large digirth and dichromatic number. Our result is an analogue of the aforementioned result of Bollobás [4] and Kostochka and Mazurova [12]. Note that the result involves the girth and not the digirth.

Theorem 2.1. *Let g and Δ be positive integers. There exists a digraph D of girth at least g, with $\Delta(D) \leq \Delta$, and $\chi(D) \geq a\Delta/\log\Delta$ for some absolute constant $a > 0$. For Δ sufficiently large we may take $a = \frac{1}{15}$.*

Proof. Our proof is in the spirit of Bollobás [4]. We may assume that Δ is sufficiently large.

Let $D = D(n,p)$ be a random digraph of order n defined as follows. For every $u, v \in V(D)$, we connect uv with probability $2p$, independently. Now we randomly (with probability $1/2$) assign an orientation to every edge that is present. Observe that D has no digons. We will use the value $p = \frac{\Delta}{4en}$, where e is the base of the natural logarithm.

Claim 1. *D has no more than Δ^g cycles of length less than g with probability at least $1 - \frac{1}{\Delta}$.*

Proof. Let N_l be the number of cycles of length l in D. Then

$$\mathbb{E}[N_l] \leq \binom{n}{l} l!(2p)^l \leq n^l(2p)^l \leq (\tfrac{\Delta}{4})^l.$$

Therefore, the expected number of cycles of length less than g is at most Δ^{g-1}. So the probability that D has more than Δ^g cycles of length less than g is at most $1/\Delta$ by Markov's inequality. $\square$

Claim 2. *There is a set A of at most $n/1000$ vertices of D such that $\Delta(D - A) \leq \Delta$ with probability at least $\frac{1}{2}$.*

Proof. Let X_d be the number of vertices of total degree d, $d = 0, 1, ..., n-1$. Following [4], define the *excess degree of* D to be $ex(D) = \sum_{d=\Delta+1}^{n-1}(d - \Delta)X_d$. Clearly, there is a set of at most $ex(D)$ arcs (or vertices) whose removal reduces the maximum total degree of D to at most Δ.

Now, we estimate the expectation of X_d. By linearity of expectation, we have:

$$
\begin{aligned}
\mathbb{E}[X_d] &\leq n\binom{n-1}{d}(2p)^d \\
&\leq n\left(\frac{e(n-1)}{d}\right)^d\left(\frac{\Delta}{2en}\right)^d \\
&\leq n\left(\frac{\Delta}{2d}\right)^d.
\end{aligned}
$$

Therefore, by linearity of expectation we have that

$$
\begin{aligned}
\mathbb{E}[ex(D)] &\leq \sum_{d=\Delta+1}^{n-1} nd\left(\frac{\Delta}{2d}\right)^d \\
&\leq \frac{n\Delta}{2}\sum_{d=\Delta+1}^{n-1}\left(\frac{\Delta}{2d}\right)^{d-1} \\
&\leq \frac{n\Delta}{2}\sum_{d=\Delta+1}^{n-1}\left(\frac{1}{2}\right)^{d-1} \\
&\leq \frac{n\Delta}{2}\cdot\frac{(\frac{1}{2})^\Delta}{1-\frac{1}{2}} \\
&= n\cdot\frac{\Delta}{2^\Delta} \\
&\leq \frac{n}{2000}.
\end{aligned}
$$

Now, by Markov's inequality, $\mathbb{P}[ex(D) > n/1000] < 1/2$. $\square$

Let $\alpha(D)$ be the size of a maximum acyclic set of vertices in D. The following result will be used in the proof of our next claim and also in Section 3.

Theorem 2.2 ([20])**.** *Let $D \in D(n,p)$. There is an absolute constant W such that if p satisfies $np \geq W$, then, a.a.s.*

$$\alpha(D) \leq \left(\frac{2}{\log q}\right)(\log np + 3e),$$

where $q = (1-p)^{-1}$.

Claim 3. *Let* $D \in D(n,p)$. *Then* $\alpha(D) \leq \frac{4en\log\Delta}{\Delta}$ *with high probability.*

Proof. Since Δ is sufficiently large, Theorem 2.2 applies and the result follows. $\square$

Now, pick a digraph D that satisfies claims 1, 2 and 3. After removing at most $n/1000 + \Delta^g \leq n/100$ vertices, the resulting digraph D^* has maximum degree at most Δ and girth at least g. Clearly, $\alpha(D^*) \leq \alpha(D)$. Therefore, $\chi(D^*) \geq \frac{n(1-1/100)}{4en\log\Delta/\Delta} \geq \frac{\Delta}{5e\log\Delta}$. $\square$

3 Another result of the same nature

A result of Erdős [5] states that there exist graphs of large chromatic number where every induced subgraph with up to a constant fraction number of the vertices is 3-colorable. In particular, it is proved that for every k there exists $\epsilon > 0$ such that for all n sufficiently large there exists a graph G of order n with $\chi(G) > k$ and yet $\chi(G[S]) \leq 3$ for every $S \subset V(G)$ with $|S| \leq \epsilon n$.

The 3-colorability in the aforementioned theorem cannot be improved. A result of Kierstead, Szemeredi and Trotter [11] (with later improvements by Nilli [17] and Jiang [9]) shows that every 4-chromatic graph of order n contains an odd cycle of length at most $8\sqrt{n}$.

We prove the following analog for digraphs. Our proof follows the ideas of Erdős found in [1].

Theorem 3.1. *For every* k, *there exists* $\epsilon > 0$ *such that for every sufficiently large integer* n *there exists a digraph* D *of order* n *with* $\chi(D) > k$ *and yet* $\chi(D[S]) \leq 2$ *for every* $S \subset V(D)$ *with* $|S| \leq \epsilon n$.

Proof. Clearly, we may assume that $\log k \geq 3$ and $k \geq \sqrt{W}$, where W is the constant in Theorem 2.2. Let us consider the random digraph $D = D(n,p)$ with $p = \frac{k^2}{n}$ and let $0 < \epsilon < k^{-5}$.

We first show that $\chi(D) > k$ with high probability. Since k is sufficiently large, Theorem 2.2 implies that $\alpha(D) \leq 6n\log k/k^2$ with high probability. Therefore, almost surely $\chi(D) \geq \frac{1}{6}k^2/\log k > k$.

Now, we show that with high probability every set of at most ϵn vertices can be colored with at most two colors. Suppose there exists a set S with $|S| \leq \epsilon n$ such that $\chi(D[S]) \geq 3$. Let $T \subset S$ be a 3-critical subset, i.e. for every $v \in T$, $\chi(D[T] - v) \leq 2$. Let $t = |T|$. Since $D[T]$ is 3-critical, every $v \in T$ satisfies $\min\{d^+_{D[T]}(v), d^-_{D[T]}(v)\} \geq 2$ for otherwise a 2-coloring

of $D[T]-v$ could be extended to $D[T]$. This implies that $D[T]$ has at least $2t$ arcs. The probability of this event is at most

$$\begin{aligned}
\sum_{3\le t\le \epsilon n}\binom{n}{t}\binom{2\binom{t}{2}}{2t}\left(\frac{k^2}{n}\right)^{2t} &\le \sum_{3\le t\le \epsilon n}\left(\frac{en}{t}\right)^t\left(\frac{et(t-1)}{2t}\right)^{2t}\left(\frac{k^2}{n}\right)^{2t}\\
&\le \sum_{3\le t\le \epsilon n}\left(\frac{e^3tk^4}{4n}\right)^t\\
&\le \epsilon n \max_{3\le t\le \epsilon n}\left(\frac{7tk^4}{n}\right)^t \qquad (1)
\end{aligned}$$

If $3\le t\le (\log n)^2$, then $(7tk^4/n)^t \le (7(\log n)^2k^4/n)^t \le \left(7(\log n)^2k^4/n\right)^3 = o(1/n)$. Similarly, if $(\log n)^2\le t\le \epsilon n$, then $\left(7tk^4/n\right)^t \le (7\epsilon k^4)^t \le (7/k)^t \le (7/k)^{(\log n)^2} = o(1/n)$.

These estimates and (1) imply that the probability that $\chi(D[S])\le 2$ is $o(1)$. This completes the proof. $\square$

The 2-colorability in the previous theorem cannot be decreased to 1 due to the following theorem.

Theorem 3.2. *If D is a digraph with $\chi(D)\ge 3$ and of order n, then it contains a directed cycle of length $o(n)$.*

Proof. In the proof we shall use the following digraph analogue of Erdős-Posa Theorem. Reed et al. [19] proved that for every integer t, there exists an integer $f(t)$ so that every digraph either has t vertex-disjoint directed cycles or a set of at most $f(t)$ vertices whose removal makes the digraph acyclic. Define $h(n)=\max\{t : tf(t)\le n\}$. It is clear that $h(n)=\omega(1)$.

Let c be the length of a shortest directed cycle in D and let $t := h(n)$. If D has t vertex-disjoint directed cycles, then $ct\le n$ which implies that $c\le \frac{n}{h(n)} = o(n)$. Otherwise, there exists a set S of vertices with $|S|\le f(t)$ such that $V(D)\backslash S$ is acyclic. Since $\chi(D)\ge 3$, we have that $\chi(D[S])\ge 2$, which implies that S contains a directed cycle of length at most $|S|\le f(t)\le \frac{n}{t} = \frac{n}{h(n)} = o(n)$. $\square$

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